Linear Algebra, Winter 2022

List 6

Eigenvalues, complex number intro

Let M be a square matrix. If
$$M\vec{v} = \lambda\vec{v}$$
with $\vec{v} \neq \vec{0}$ then the vector $\vec{v}$ is called an **eigenvector** of M and the number λ is called an **eigenvalue** of M.

- The eigenvalues of M are exactly the numbers λ for which
$$\det(A - \lambda I) = 0.$$
- The determinant of M is exactly equal to the product of all its eigenvalues.

138. Find an eigenvector of $\begin{bmatrix} 7 & -5 \\ 0 & 8 \end{bmatrix}$ corresponding to the eigenvalue 8. That is, find a non-zero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $\begin{bmatrix} 7 & -5 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 8\begin{bmatrix} x \\ y \end{bmatrix}$.

$\boxed{\begin{bmatrix} -5 \\ 1 \end{bmatrix}}$ or any multiple of that

139. Find the eigenvalues of $\begin{bmatrix} 4 & 1 \\ -2 & 8 \end{bmatrix}$.

$\det\left(\begin{bmatrix} 4-\lambda & 1 \\ -2 & 8-\lambda \end{bmatrix}\right) = (4-\lambda)(8-\lambda) - (1)(-2) = \lambda^2 - 12\lambda + 34.$

The roots of $\lambda^2 - 12\lambda + 34$ are $\boxed{6 + \sqrt{2} \text{ and } 6 - \sqrt{2}}$.

140. (a) Find the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$. $\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 7 & 8-\lambda \end{bmatrix}\right) = 0$ gives $\lambda^2 - 10\lambda + 9 = 0$, so the eigenvalues are $\boxed{9 \text{ and } 1}$.

(b) Find the eigenvectors of $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$. For $\lambda = 9$, we want $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 9\begin{bmatrix} x \\ y \end{bmatrix}$, so $\begin{bmatrix} 2x+y \\ 7x+8y \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \end{bmatrix}$. The solutions to $\begin{cases} 2x+y = 9x \\ 7x+8y = 9y \end{cases}$ are $\boxed{\text{any multiple of } \begin{bmatrix} 1 \\ 7 \end{bmatrix}}$.

For $\lambda = 1$ we get $\boxed{\text{any multiple of } \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

141. (a) Find the eigenvalues of $\begin{bmatrix} 4 & 1 \\ -8 & 8 \end{bmatrix}$.

$\det\left(\begin{bmatrix} 4-\lambda & 1 \\ -8 & 8-\lambda \end{bmatrix}\right) = (4-\lambda)(8-\lambda) - (1)(-8) = \lambda^2 - 12\lambda + 40.$

The zeros of $\lambda^2 - 12\lambda + 40$ are $\boxed{6 + 2i \text{ and } 6 - 2i}$.

☆(b) Find the eigenvectors of $\begin{bmatrix} 4 & 1 \\ -8 & 8 \end{bmatrix}$.

$\boxed{\text{any multiple of } \begin{bmatrix} 1+i \\ 4 \end{bmatrix}}$ and $\boxed{\text{any multiple of } \begin{bmatrix} 1-i \\ 4 \end{bmatrix}}$

142. Find the eigenvalues of $\begin{bmatrix} 4 & 8 & 1 \\ 7 & 14 & 6 \\ 1 & 2 & 6 \end{bmatrix}$. $\boxed{0 \text{ and } 5 \text{ and } 19}$

143. If the matrix M satisfies

$$M\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ -6 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \end{bmatrix},$$

find the eigenvalues of M.

The middle equation is irrelevant because $[24, -6]$ is not parallel to $[3, 3]$. The other two equations show that $\boxed{-6 \text{ and } 3}$ are eigenvalues because $[-12, 12] = (-6)[2, -2]$ and $[21, 6] = 3[7, 2]$.

144. Calculate the determinant of a 3×3 matrix whose eigenvalues are 19, 1, and -5. Determinant = product of eigenvalues = $\boxed{-95}$.

145. Expand $(2 + 3a)(5 - 4a)$. $\boxed{10 + 7a - 12a^2}$

146. Re-write $(1+t)(8+3t)$ in the form __ + __t if $t^2 = 10$. $8 + 11t + 3t^2 = \boxed{38 + 11t}$

$$i^2 = -1$$

147. Re-write $(1 + i)(8 + 3i)$ in the form __ + __i, knowing that $i^2 = -1$.
$8 + 11i + 3i^2 = \boxed{5 + 11i}$

148. Give the determinant of a matrix whose eigenvalues are...
(a) 4, 3, and 0. $\boxed{0}$ (b) $1 - 4i$, $1 + 4i$, and 2. $(1 - 4i)(1 + 4i)(2) = \boxed{34}$
(c) 9 and $\frac{1}{4}$. $\boxed{2.25}$

149. Simplify each of the following: $i^2 = \boxed{-1}$, $i^3 = \boxed{-i}$, $i^4 = \boxed{1}$, $i^5 = \boxed{i}$, $i^{15} = \boxed{-i}$, $i^{202} = \boxed{-1}$, $i^{1285100} = \boxed{1}$, $i^{-1} = \boxed{-i}$.

150. Write the following in the form $a + bi$, where a and b are real numbers.

(a) $(-6 + 5i) + (2 - 4i) = \boxed{-4 + i}$

(b) $(1 + 2i)(2 + 3i) = \boxed{-4 + 7i}$

(c) $(-5 + 2i) - (2 - i) = \boxed{-7 + 3i}$

(d) $(2 - 3i)(2 + 3i) = \boxed{13}$

(e) $(1 + i)(2 - i)(3 + 2i) = \boxed{7 + 9i}$

(f) $(1 - 2i)^3 = \boxed{-11 + 2i}$

(g) $(-2i)^6 = \boxed{-64}$

(h) $(1 + i)^4 = \boxed{-4}$

151. Write $\dfrac{1+2i}{2-3i}$ in the form $a+bi$. (Hint: $\dfrac{1+2i}{2-3i} \times \dfrac{2+3i}{2+3i}$.)

Using 150(b) and 150(d), $\dfrac{(1+2i)(2+3i)}{(2-3i)(2+3i)} = \dfrac{-4+7i}{13} = \boxed{\dfrac{-4}{13} + \dfrac{7}{13}i}$

> If $z = a+bi$, where a and b are real numbers, then the **real part** of z is a, and the **imaginary part** of z is b (not bi).
>
> The **magnitude** (or **modulus**) of $z = a+bi$ is the distance between $(0,0)$ and (a,b) on an xy-plane; it is written as $|z|$. The **argument** of z is the angle between the positive x-axis and the line from $(0,0)$ to (a,b); it is written as $\arg(z)$.

152. What is the real part of $(5+6i)(2i)$? $\boxed{-12}$

153. (a) Calculate the length of the hypotenuse of a right triangle whose legs have lengths 1 and $\sqrt{2}$.

(b) Calculate the distance between the points $(0,0)$ and $(1,\sqrt{2})$.

(c) Calculate the magnitude of the vector $[1,\sqrt{2}]$, often written $\big|[1,\sqrt{2}]\big|$.

(d) Calculate the magnitude of the complex number $1+\sqrt{2}i$, often written $\big|1+\sqrt{2}i\big|$. Every part is the same! Answer: $\boxed{3}$

154. Compute $\big|2+7i\big|$. $\boxed{\sqrt{53}}$

155. What is the magnitude of $\sqrt{11}\cos(\pi/8) + \sqrt{11}\sin(\pi/8)i$? $\boxed{\sqrt{11}}$

156. (a) What is the real part of $1-\sqrt{3}\,i$? $\boxed{1}$

(b) What is the imaginary part of $1-\sqrt{3}\,i$? $\boxed{-\sqrt{3}}$ Note: *not* $-\sqrt{3}\,i$.

(c) Compute $|1-\sqrt{3}\,i|$. $\boxed{2}$ (d) Compute $\arg(1-\sqrt{3}\,i)$. $\boxed{-\pi/3}$

(e) Give values for r and θ such that $1-\sqrt{3}\,i = r\big(\cos(\theta)+i\sin(\theta)\big)$. $\boxed{r=2,\ \theta=-\pi/3}$

157. Calculate each of the following:

(a) the real part of $2i-7$. $\boxed{-7}$

(b) the imaginary part of $(3+2i)(5i)$. $\boxed{15}$

(c) the imaginary part of 4. $\boxed{0}$

(d) the imaginary part of i^2 $\boxed{0}$

(e) the real part of i^2 $\boxed{-1}$

(f) $\arg(-3i)$. $\boxed{-90^\circ \text{ or } -\frac{1}{2}\pi}$ since we usually use $-\pi < \arg(z) \le \pi$

(g) $\arg(5+5i)$. $\boxed{45^\circ \text{ or } \frac{1}{4}\pi}$

(h) $\arg(5-5i)$. $\boxed{-45^\circ \text{ or } -\frac{1}{2}\pi}$

Rectangular form: $a+b\,i$, or $a+i\,b$, or $b\,i+a$, or similar, where a and b are real numbers and usually are simplified. If a is zero, you can skip writing "0+"..., and if $b=0$ you can skip writing ..."$+0i$".

Polar form: $r\cos(\theta)+r\sin(\theta)\,i$, or $r\big(\cos\theta+i\sin\theta\big)$, or similar. Requires $r\geq 0$.

158. Re-write $10\cos(-\frac{\pi}{4})+10\sin(-\frac{\pi}{4})i$ without trig functions. $\boxed{\frac{10}{\sqrt{2}}-\frac{10}{\sqrt{2}}i}$

159. Write each of the following in polar form.

That is, write each as $\_\_\big(\cos(\_\_)+i\sin(\_\_)\big)$, where the angles for cosine and sine must be equal and the first blank must be positive.

(a) $1-\sqrt{3}i = \boxed{2\cos(-\frac{\pi}{3})+2\sin(-\frac{\pi}{3})i}$

(b) $-\sqrt{5}+\sqrt{15}\,i = \boxed{\sqrt{20}\big(\cos(\frac{3}{4}\pi)+i\sin(\frac{3}{4}\pi)\big)}$, see **Task 3** from List 1

(c) $3+3i = \boxed{3\sqrt{2}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})}$

(d) $-3i = \boxed{3(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})}$

(e) $1+\sqrt{3}\,i = \boxed{2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})}$

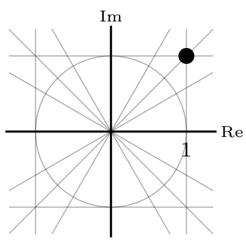
(f) $2-2\sqrt{3}\,i = \boxed{4(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3})}$

(g) $\frac{\sqrt{3}-i}{7} = \boxed{\frac{2}{7}(\cos\frac{-\pi}{6}+i\sin\frac{-\pi}{6})}$

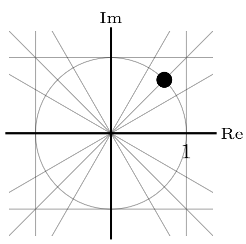
(h) $\sqrt{-1} = \boxed{1(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})}$

160. Write $2+\sqrt{2}\cos(3\pi/4)+\sqrt{2}\sin(3\pi/4)i$ in both rectangular and polar form. $\boxed{1+i=\sqrt{2}\cos(\pi/4)+\sqrt{2}\sin(\pi/4)i}$

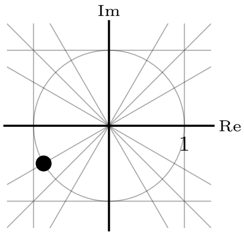
161. Write each number below in both rectangular and polar form.

(a) 

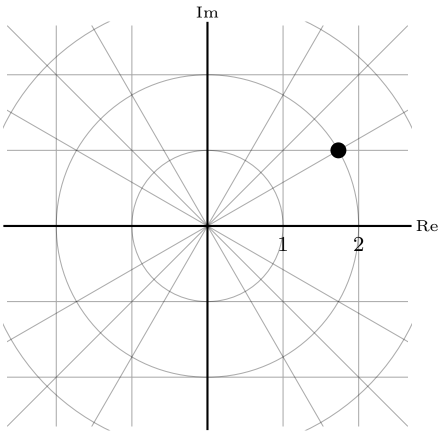
$\boxed{1+i=\sqrt{2}\big(\cos(\pi/4)+i\sin(\pi/4)\big)}$

(b) 

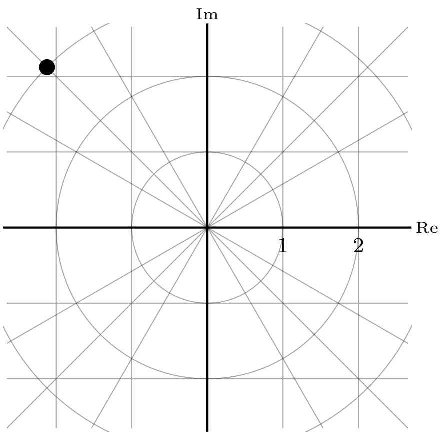
$\boxed{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\cos(\pi/4)+i\sin(\pi/4)}$

(c) 

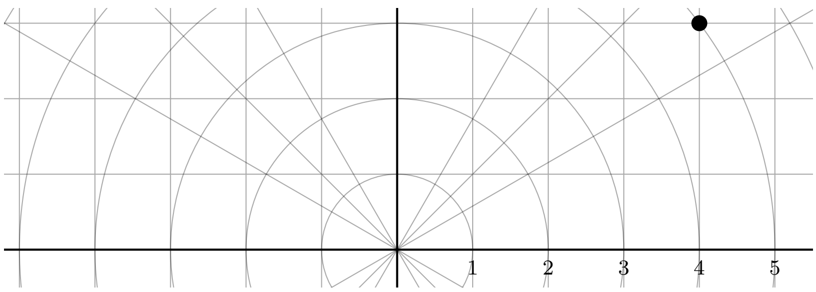
$\cos(7\pi/6) + i\sin(7\pi/6) = \cos(-5\pi/6) + i\sin(-5\pi/6) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(d) 

$2\big(\cos(\pi/6) + i\sin(\pi/6)\big) = \sqrt{3} + i$

(e) 

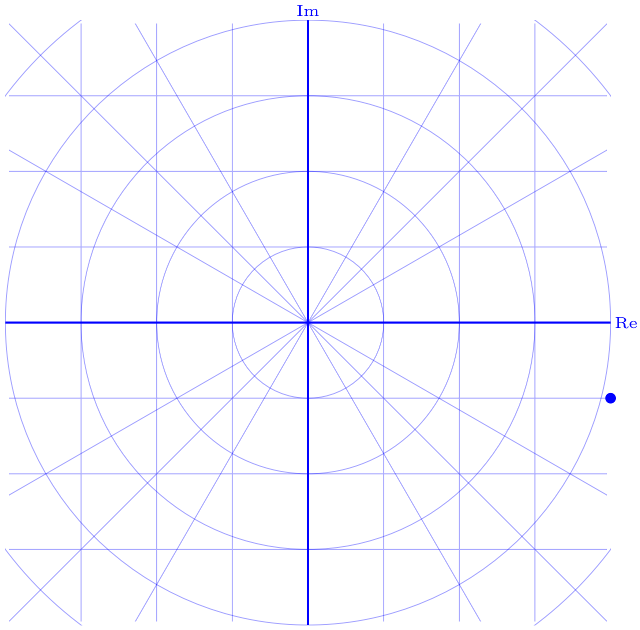
$3\big(\cos(3\pi/4) + i\sin(3\pi/4)\big) = \frac{3}{\sqrt{2}} + \frac{-3}{\sqrt{2}}i$

(f) 

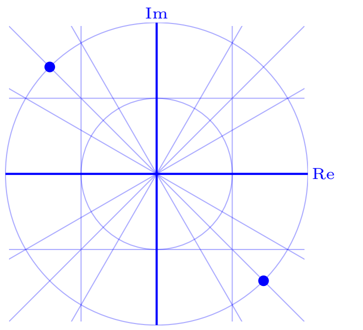
$4 + 3i = 5\big(\cos(\tan^{-1}\frac{3}{4}) + i\sin(\tan^{-1}\frac{3}{4})\big)$

162. On a complex plane, *draw* the number(s)...

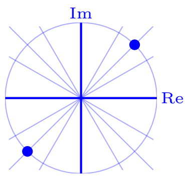
(a) $4 - i$



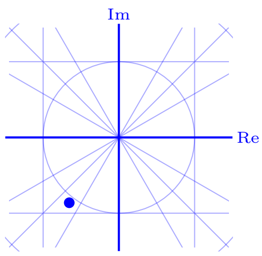
(b) $\sqrt{2} - \sqrt{2}i$ and $-\sqrt{2} + \sqrt{2}i$



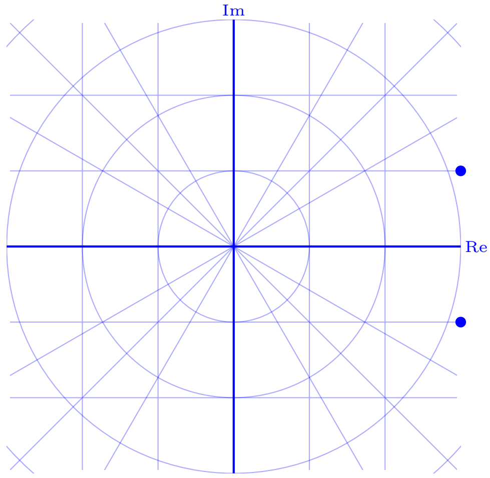
(c) $\dfrac{1+i}{\sqrt{2}}$ and $\dfrac{-1-i}{\sqrt{2}}$



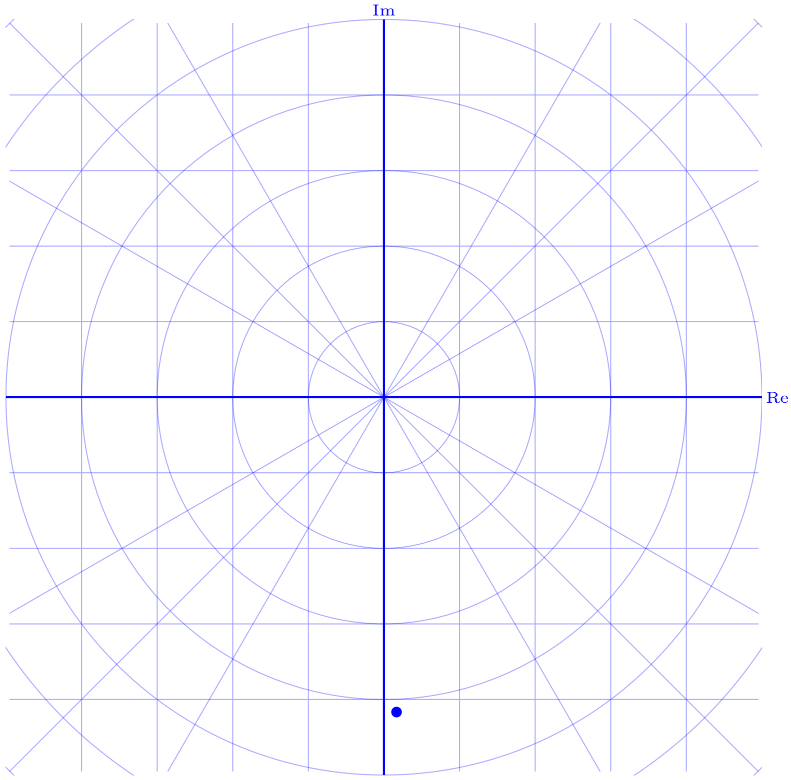
(d) $-\dfrac{19}{29} - \dfrac{25}{29}i$



(e) $3 - i$ and $3 + i$



(f) $\dfrac{1 - 25i}{6}$



163. Re-write $(q\, n^{st})^3$ in the form $\_\_\, n^{\_\_ t}$. $q^3\, n^{3st}$

164. Re-write $(r\, e^{i\theta})^3$ in the form $\_\_\, e^{\_\_ i}$. $r^3\, e^{3\theta\, i}$